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Heisenberg scaling precision in multi-mode distributed quantum metrology

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Abstract

We consider the estimation of an arbitrary parameter φ, such as the temperature or a magnetic field, affecting in a distributed manner the components of an arbitrary linear optical passive network, such as an integrated chip. We demonstrate that Heisenberg scaling precision (i.e. of the order of $1/N$, where N is the number of probe photons) can be achieved without any iterative adaptation of the interferometer hardware and by using only a simple, single, squeezed light source and well-established homodyne measurements techniques. Furthermore, no constraint on the possible values of the parameter is needed but only a preliminary shot-noise estimation (i.e. with a precision of $\sqrt{N}$) easily achievable without any quantum resources. Indeed, such a classical knowledge of the parameter is enough to prepare a single, suitable optical stage either at the input or the output of the network to monitor with Heisenberg-limited precision any variation of the parameter to the order of $1/\sqrt{N}$ without the need to iteratively modify such a stage.

1. Introduction

Due to the discreteness of all natural phenomena, the error in the estimation of a physical parameter φ through a measurement employing N probes (e.g. photons, electrons) is strongly limited by the so-called 'shot noise' factor of $1/\sqrt{N}$. However, it has been proven that quantum features such as entanglement and squeezing can be exploited to go beyond the shot-noise limit and reach a precision of order $1/N$, which is the so-called Heisenberg limit [1–9].

The situation most commonly considered in quantum metrology is the estimation of an optical phase [1, 3, 5, 7, 10–12] or a phase-like parameter [4, 6, 13], that, e.g., is encoded through a unitary evolution generated by a φ-independent Hermitian operator. Considerable effort has recently been done in the direction of distributed quantum metrology in the multiparameter setting for the estimation of particular functions of multiple parameters encoded in a specific manner to spatially separated nodes [14–20]. A situation which has not been considered in these 'distributed' scenarios is the estimation of a single parameter encoded in an arbitrary manner in different components of a multi-mode interferometer (see figure 1). This could be the case for the estimation of the magnitude of an external field through its influence on the optical properties of the components of an arbitrary interferometer. For example, temperature has been used to tune the reflection and transmission coefficients of beamsplitters in on-chip interferometers [21, 22]. Temperature can also be used to change the optical path length through a material of index of refraction. Thus, the effects of one parameter (temperature in this case) are distributed across a network that consists of beam splitters and phase shifters. Such general encodings of φ into an arbitrary

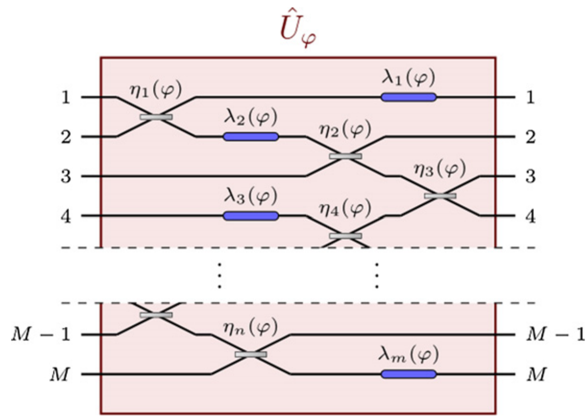

Figure 1. An M-mode passive linear interferometer composed of an arbitrary number of beam splitters and phase shifters having different and generic dependencies on a parameter φ. This scheme may represent for example the situation where φ is the temperature or the magnitude of an electromagnetic or a gravitational field affecting the optical properties of the interferometer components.

passive linear interferometer give rise to parameter-dependent generators, which has only recently started to be considered [23]. Unfortunately, the current challenge in this general scenario is that the estimation protocol becomes highly adaptive, since *both* the optimal input state of the probe, *and* the optimal measurement to be performed depend on the unknown value of φ [23]. This challenge is also relevant in the context of distributed quantum metrology with multiple unknown parameters and has been overcome only if constraints in the range of variation of the parameters are given [14, 15, 18, 19, 24, 25]. Can such a serious drawback be overcome in order to ultimately implement quantum technologies for distributed quantum-enhanced metrology?

In this work, we show that this is possible by introducing an experimentally feasible metrological setup achieving, when appropriate classical post-processing data analysis is performed, Heisenberg scaling in the estimation of a generic parameter distributed over an arbitrary passive linear optical interferometer. Independent of the structure of the multi-mode interferometer, we always pick a single mode squeezed vacuum as the only input probe and balanced homodyne detection as measurement. No constraint on the values the parameter can assume is needed: instead, only a preliminary classical estimation, namely with shot-noise limited precision, suffices to correctly prepare our setup. In fact, only a single additional passive linear optical stage is needed either at the input or at the output of the network, and its preparation only requires a 'classical' knowledge of the parameter φ to estimate. For simplicity and without losing generality, we will consider the case where such a stage $\hat{V}_{\text{out}}$ is placed at the output of the network as in figure 2. Remarkably, we will show that Heisenberg limited sensitivity can be obtained independently of any parameter-independent passive linear optical stage $\hat{V}_{\text{in}}$ one can place at the input. This includes the case where $\hat{V}_{\text{in}}$ is the identity operator, i.e. in the absence of any input stage. The role of $\hat{V}_{\text{in}}$ and $\hat{V}_{\text{out}}$ as parameter-independent and parameter-dependent stages can be inverted without any substantial change in the following of this work. Noticeably, our setup is experimentally feasible since it employs only Gaussian states and measurements, which are easier to manipulate and implement with respect to other states and measurements commonly used in literature. We will show that all the information on the unknown parameter is encoded in the variance of the quadrature field we measure through homodyne detection. The dependence of the variance on the parameter is non-linear, which makes the search of an efficient and unbiased estimator an hard task [26, 27]. However, it is well known that, also in non-linear models, the maximum-likelihood estimator is an asymptotically efficient estimator [27–30]. Then, the use of the maximum-likelihood estimator, coupled with our metrological setup, guarantees in the asymptotic regime the saturation of the Fisher information, and thus the Heisenberg-scaling sensitivity.

2. Setup

Let us consider a given M-channel passive linear interferometer which depends on the parameter φ to be estimated. The action of the interferometer on the input states is described by a passive linear unitary $\hat{U}_\varphi$.

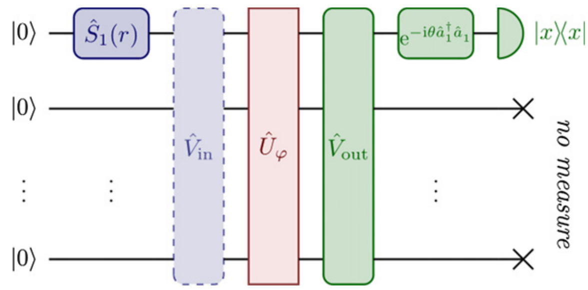

Figure 2. Block diagram of the proposed setup. The parameter φ to be estimates is encoded in any given passive linear optical network $\hat{U}_\varphi$. The input state is obtained by squeezing the vacuum in a single mode with N photons in average, while at the output of the setup the quadrature field x_θ is measured by means of homodyne detection, and the phase of the local oscillator shall be tuned to a value $\theta = \theta_\varphi$ which satisfies condition (5). A single auxiliary stage is needed at the input or the output of $\hat{U}_\varphi$ (e.g. $\hat{V}_{\rm out}$ at the output of the network $\hat{U}_\varphi$), whose preparation requires only a 'classical', namely shot-noise precise, prior estimation of φ, such that condition (6) is verified. We show that Heisenberg-scaling sensitivity can be achieved for an arbitrary choice of a parameter-independent stage, in the figure depicted as $\hat{V}_{\rm in}$ at the input of the network $\hat{U}_\varphi$.

The preparation of the input probe consists in the injection of a single-mode squeezed vacuum state in the first port of a unitary stage $\hat{V}_{\rm in}$, which is used to scatter the photons injected among all the modes. The input state in our protocol is therefore given by $|\psi\rangle = \hat{V}_{\rm in}\hat{S}_1(r)|{\rm vac}\rangle$, where $\hat{S}_1(r) = \mathrm{e}^{\frac{r}{2}(\hat{a}_1^2 - \hat{a}_1^{\dagger 2})}$ is the single-mode squeezing operator with squeezing parameter $r > 0$, and $|{\rm vac}\rangle$ is the M-channel vacuum state. The average number of photons injected in the apparatus is thus $N = \sinh^2 r$. At the output of the interferometer, the unitary $\hat{V}_{\rm out}$ is applied in order to refocus all the photons into a single mode, namely the first one, in order to capture all the information about the parameter in a single channel. In such a way the estimation can be carried out with a single homodyne detection performed on the aforementioned channel. For a linear passive unitary $\hat{U}$, we denote by U the $M \times M$ unitary matrix defined by

$$\hat{U}^\dagger \hat{a}_j \hat{U} = \sum_{k=1}^{M} U_{jk}\hat{a}_k, \tag{1}$$

whose elements are the single photon transition amplitudes. Then, the probability that a photon injected in the first port of $\hat{V}_{\rm in}$ comes out from the first port of $\hat{V}_{\rm out}$ is given by

$$P_\varphi = |(V_{\rm out}U_\varphi V_{\rm in})_{11}|^2, \tag{2}$$

so that if the refocusing procedure is not perfect there will be some probability of photons scattering into other channels, which is quantified by $1 - P_\varphi$. Ideally, we would like to exploit the information encoded by the interferometric evolution in all the photons within the injected squeezed state. This corresponds to the condition $P_\varphi = 1$: we are essentially channelling all the information about the parameter in a single output channel, namely the first one. Then, a homodyne detection of the field quadrature $\hat{x}_\theta$ is performed on the first channel, where θ is the reference phase of the local oscillator employed to perform the measurement. Let

$$\gamma_\varphi = \arg[(V_{\rm out}U_\varphi V_{\rm in})_{11}], \tag{3}$$

be the phase accumulated through the whole setup by the field at the first output port, which will be assumed such that $\partial_\varphi\gamma_\varphi \neq 0$. The latter assumption means that the phase γ_φ is not constant around the value φ of the parameter, which is instead effectively encoded in γ_φ, as a small variation of φ implies a proportional variation of γ_φ. For those cases in which $\partial_\varphi\gamma_\varphi = 0$, a different suitable choice of $\hat{V}_{\rm in}$ would restore the sensitivity of the setup. Noticeably, this would happen for a typical $\hat{V}_{\rm in}$ [31]. The squeezed direction of the probe at the output will be $\gamma_\varphi \pm \pi/2$, so that the minimum uncertainty quadrature field is $\hat{x}_{\gamma_\varphi + \pi/2}$.

3. Heisenberg scaling

The ultimate precision $\delta\varphi$ achievable in a given estimation procedure based on ν measurements is determined by the Fisher information $F(\varphi)$ through the Cramer–Rao bound [28]:

$$\delta\varphi \geqslant \frac{1}{\sqrt{\nu F(\varphi)}}. \tag{4}$$

Evaluating $F(\varphi)$ associated with the described setup, we find that the Heisenberg scaling can be asymptotically achieved for large N if the following conditions are satisfied (see appendix A)

$$\theta_\varphi \sim \gamma_\varphi \pm \frac{\pi}{2} + \frac{k_\varphi}{N}, \tag{5}$$

$$P_\varphi \sim 1 - \frac{\ell_\varphi}{N}, \tag{6}$$

where $\ell_\varphi \geqslant 0$ and $k_\varphi \neq 0$ are arbitrary but both independent of N, and where θ_φ is an optimal choice for θ. From a physical point of view, ℓ_φ represents the average number of photons scattered into channels which are not measured, while k_φ/N represents the 'resolution' needed in the homodyne detection. In practice, one can even fix k_φ to a constant value without using additional resources.

Under conditions (5) and (6), the Fisher information asymptotically reads (see appendix A)

$$F(\varphi) \equiv \frac{1}{2}\left(\frac{\partial_\varphi \Delta_\varphi}{\Delta_\varphi}\right)^2 \sim 8\varrho(k_\varphi, \ell_\varphi)(\partial_\varphi \gamma_\varphi)^2 N^2, \tag{7}$$

where

$$\Delta_\varphi = \frac{1}{2}\left[1 + P_\varphi\left(2\sinh^2(r) + \cos[2(\gamma_\varphi - \theta)]\sinh\ 2r\right)\right], \tag{8}$$

is the variance of the measured quadrature, and $\varrho(k,\ell) = [8k/(16k^2 + 4\ell + 1)]^2$. Then, according to the Cramer–Rao bound (4) the ultimate precision achievable with this setup is given by $\delta\varphi = O(1/N)$. The search for an unbiased efficient estimator, i.e. saturating the inequality (4), is complicated by the fact that the measurement results depend on the parameter only through the variance (8) which has a nonlinear dependence on φ [26]. However, it is known that the bound can be asymptotically saturated through post-processing data analysis employing the maximum-likelihood estimator [27–30]. The prefactor $\varrho(k,\ell)$ reaches its maximum value $\varrho = 1$ at $k = \pm 1/4$ and $\ell = 0$, while it vanishes at $k = 0$, hence the requirement $k_\varphi \neq 0$ needed to reach Heisenberg scaling. At $k = 0$ the quadrature field being measured has the minimal variance, so a vanishing Fisher information for this value of k may appear counter-intuitive. However, this occurs as a consequence of the fact that, for a local-oscillator phase $\theta = \gamma_\varphi \pm \pi/2$, the probability distribution based on a homodyne measurement in the first output channel is locally insensitive to variations of φ. Indeed, when condition (6) holds, the output state in the first channel is essentially a vacuum squeezed state rotated by the phase γ_φ in (3) accumulated through the interferometer. More precisely, the probability distribution depends only on the variance of $\hat{x}_\theta$, which has a minimum for this value of θ, hence being a stationary point.

The aforementioned conditions (5) and (6) imply an adaptive procedure, since they depend on the true value of the unknown parameter φ. However, condition (5) only establishes the minimal resolution required in the variation of θ during the feedback procedure of the homodyne detection, and, quite interestingly, condition (6) can be satisfied by manipulating only one of the two unitary stages, while leaving the other one arbitrary. The Heisenberg sensitivity is preserved even if a ratio ℓ_φ/N of the photons in the squeezed input probe is not detected in the first channel, meaning that our protocol is robust against imperfections in the optimized stage. This stage can thus be efficiently built even if the prior knowledge of φ is affected by some error $\delta\varphi$.

Noticeably, this uncertainty $\delta\varphi$ is allowed to be of the order $1/\sqrt{N}$ to satisfy condition (6). Hence, a classical estimation of φ (i.e. shot-noise limited in the number of resources) is sufficient to gather the information needed to prepare the parameter-dependent stage $\hat{V}_{\text{out}}$ for an arbitrary parameter-independent stage $\hat{V}_{\text{in}}$. This result is due to the very structure of $P_\varphi = |(V_{\text{out}} U_\varphi V_{\text{in}})_{11}|^2$, which is essentially nothing but a transition probability $P = |\langle v_{\text{out}}|v_{\text{in}}\rangle|^2$ between $|v_{\text{in}}\rangle = U_\varphi V_{\text{in}}|e_1\rangle$ and $|v_{\text{out}}\rangle = V_{\text{out}}^\dagger|e_1\rangle$, with $|e_1\rangle = (1, 0, \ldots, 0)^{\mathrm{T}}$. A simple geometrical consequence of this expression is that a small tilt of order $O(1/\sqrt{N})$ between the unit vectors $|v_{\text{in}}\rangle$ and $|v_{\text{out}}\rangle$ yields a quadratic reduction of their transition probability

$$P = \cos^2\left(O\left(\frac{1}{\sqrt{N}}\right)\right) \sim 1 - O\left(\frac{1}{N}\right). \tag{9}$$

Furthermore, given that for any unknown parameter φ no prior knowledge of its value is required with higher precision than $1/\sqrt{N}$, the adapted interferometer is also able to monitor the value of the parameter in an overall interval of the same order with Heisenberg-limited precision without any further change of the adapted stage.

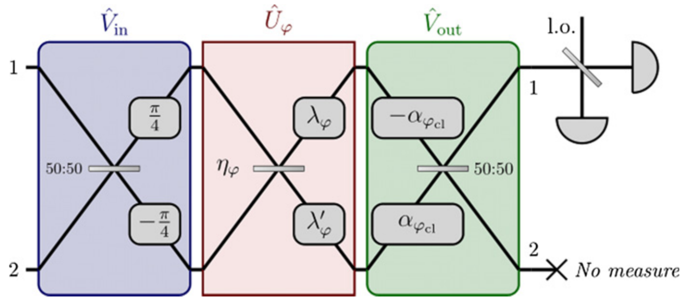

Figure 3. A two-channel example. The parameter φ is encoded into the reflectivity $\sin\eta_\varphi$ of a beam splitter, and into the phase shifts λ_φ and λ'_φ associated with its two arms. A non-adapted choice of $\hat{V}_{\rm in}$ is shown in the figure, realized with a beam splitter and two $\pm\frac{\pi}{4}$-phase shifts not depending on φ. The adaptation here is performed only on $\hat{V}_{\rm out}$ through the tuning of $\alpha_{\varphi_{\rm cl}} = (\lambda_{\varphi_{\rm cl}} - \lambda'_{\varphi_{\rm cl}})/2 - \pi/4$, where $\varphi_{\rm cl}$ is a prior classical estimation of φ. Eventually, homodyne detection is performed on the first output channel, with the local oscillator (l.o.) phase θ_φ shown in (13).

4. Example

We show how our results, which are valid for an arbitrary M-port interferometer, can be applied to a particular example of a parameter φ distributed over a two-channel interferometer, shown in the red box in figure 3. In this setup, both the reflectivity $\sin\eta_\varphi$ of a beam splitter, and the optical path lengths λ_φ and λ'_φ in the two arms depend on the parameter to be estimated: we can think of the parameter φ as the magnitude of an external field, or of a characteristic of the environment, say the temperature, which in turn influences the optical properties of the devices. The functional dependence of η_φ, λ_φ and λ'_φ on φ is assumed to be smooth. The distributed nature of φ prevents us from thinking of it as a generalized phase, a case commonly studied in literature. The unitary matrix describing phase shifts λ and λ' on the two arms is the 2×2 diagonal matrix $U_{\rm PS}(\lambda,\lambda') = \operatorname{diag}(e^{i\lambda}, e^{i\lambda'})$, while the action of a beam splitter with reflectivity $\sin\eta$ is given by $U_{\rm BS}(\eta) = e^{i\eta\sigma_y}$, with σ_y being the second Pauli matrix. Thus, the interferometer in figure 3 is described by $U_\varphi = U_{\rm PS}(\lambda_\varphi,\lambda'_\varphi)U_{\rm BS}(\eta_\varphi)$.

As previously discussed, Heisenberg scaling can be achieved by suitably adapting one of the two passive linear optical stages $\hat{V}_{\rm in}$ and $\hat{V}_{\rm out}$. Condition (6) is satisfied here with the arbitrary choice of $\hat{V}_{\rm in}$ which is shown in figure 3. It consists of a balanced beam splitter, followed by two $\pm\frac{\pi}{4}$-phase shifts, one on each arm, and thus is described by the unitary matrix $V_{\rm in} = U_{\rm PS}(\pi/4, -\pi/4)U_{\rm BS}(\pi/4)$. The stage $\hat{V}_{\rm out}$, which will have to be adapted, consists of two phase shifts, $\mp\alpha$, followed by another balanced beam splitter, and corresponds to the unitary matrix $V_{\rm out} = U_{\rm BS}(\pi/4)U_{\rm PS}(-\alpha, +\alpha)$.

A direct computation of the matrix element $(V_{\rm out}U_\varphi V_{\rm in})_{11}$ gives for this scheme the probability (2),

$$P_\varphi = \frac{1}{2}\left(1 + \sin(\lambda_\varphi - \lambda'_\varphi - 2\alpha)\right), \tag{10}$$

and the accumulated phase (3),

$$\gamma_\varphi = \frac{\lambda_\varphi + \lambda'_\varphi}{2} + \eta_\varphi + \frac{\pi}{2}. \tag{11}$$

The adaptive procedure in this example can be accomplished by simply tuning the phase shifts $\pm\alpha$ (see figure 3) to $\pm\alpha_\varphi$, with $\alpha_\varphi = (\lambda_\varphi - \lambda'_\varphi)/2 - \pi/4$, so that $P_\varphi = 1$.

Of course, tuning α requires a prior knowledge of the parameter we want to estimate. However, as discussed above, for any arbitrary given network U_φ, by denoting with $\delta\varphi = \varphi_{\rm cl} - \varphi$ the difference between a previous coarse estimation $\varphi_{\rm cl}$ and the true value φ of the parameter, a precision $\delta\varphi = O(1/\sqrt{N})$ is sufficient to reach Heisenberg scaling. Indeed, by tuning the phase shifters in the output stage according to the coarse estimation of the parameter, equation (10) reads $P_\varphi = [1 + \cos(\lambda_\varphi - \lambda_{\varphi_{\rm cl}} - \lambda'_\varphi + \lambda'_{\varphi_{\rm cl}})]/2$. Thus, a Taylor expansion for small values of $\delta\varphi$ shows that

$$P_\varphi \sim 1 - \frac{1}{4}\left(\frac{\partial(\lambda_\varphi - \lambda'_\varphi)}{\partial\varphi}\right)^2 \delta\varphi^2. \tag{12}$$

It is clear from this expression that it is possible to satisfy equation (6) with $\delta\varphi = O(1/\sqrt{N})$, which is achievable with a classical strategy employing αN photons for a measurement at the shot-noise limit, and $(1-\alpha)N$ photons for the homodyne estimation.

Finally, in accordance with equations (5) and (11), the phase θ of the local oscillator in the homodyne detection must then be tuned according to the value

$$\theta_\varphi \sim \frac{\lambda_\varphi + \lambda'_\varphi}{2} + \eta_\varphi + \frac{k_\varphi}{N}. \tag{13}$$

We notice that, although not appearing in $\hat{V}_{\text{out}}$, the value of the unknown reflectivity $\sin \eta_\varphi$ influences the quadrature field to be measured.

5. Conclusions

We provided an experimentally feasible metrological setup for the estimation of a generic parameter encoded into an M-mode passive linear interferometer with Heisenberg scaling precision. Our proposal could find applications in those situations where the parameter is distributed among different components of the interferometer, and as an example it can provide an advantageous paradigm in quantum thermometry [32] or quantum magnetometry [33, 34], in those situations where an external field such as the temperature or a magnetic field influences the optical properties of beam splitters and phase shifts in an interferometer. The practical advantages of our scheme are twofold: on one hand, it employs only a Gaussian state and Gaussian measurements, whose experimental feasibility is widely known; on the other hand, the adaptive procedure is facilitated by reducing the adaptivity to a single modification of only one interferometric stage (either at the input or the output) and with no need to iteratively change the optical hardware. Moreover, our setup does not require a perfect refocusing, being robust against loss of photons. As a matter of fact, analyzing the Cramér–Rao bound associated with our setup, we showed that the amount of information on the parameter required to prepare the optimized interferometric stage is achievable with only a classical shot-noise limited estimation. Finally, no further parameter-dependent adaptation is necessary to measure with Heisenberg-limited sensitivity any value of the parameter within an overall range of variation of the order of $1/\sqrt{N}$. Our results also motivate future studies of more specific bounds for the precision of the estimation given the Gaussian statistic of the outcome of the measurements in our setup [35]. To the best of our knowledge, our estimation setup for a distributed parameter is the first one requiring a feasible input states for the probe, and such an efficient optical hardware preparation.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Derivation of (7)

In this appendix, we outline the key steps required to derive the expression for the Fisher information shown in (7), when conditions (5) and (6) are met. Homodyne detection is performed in order to measure the quadrature field [36]

$$\hat{x}_\theta = \mathrm{e}^{\mathrm{i}\theta \hat{a}_1^\dagger \hat{a}_1} \hat{x}_1 \, \mathrm{e}^{-\mathrm{i}\theta \hat{a}_1^\dagger \hat{a}_1}, \tag{A1}$$

where $\hat{x}_1 = (\hat{a}_1 + \hat{a}_1^\dagger)/\sqrt{2}$ is the position quadrature of the field in the first mode. The POVM elements describing the measurement are given by

$$\hat{\Pi}_x = \mathrm{e}^{\mathrm{i}\theta \hat{a}_1^\dagger \hat{a}_1} |x\rangle_{11}\langle x| \mathrm{e}^{-\mathrm{i}\theta \hat{a}_1^\dagger \hat{a}_1}, \tag{A2}$$

where $\hat{x}_1|x\rangle_1 = x|x\rangle_1$. The probability of obtaining a value x from a measurement of $\hat{x}_\theta$ is determined according to the Born rule by

$$p(x|\varphi) = \mathrm{Tr}(\hat{\Pi}_x \hat{u}_\varphi \hat{S}_1(r)|\text{vac}\rangle\langle\text{vac}|\hat{S}_1^\dagger(r)\hat{u}_\varphi^\dagger). \tag{A3}$$

A calculation of (A3) can be carried out using the phase-space formalism [31], which leads to a Gaussian probability distribution:

$$p(x|\varphi) = \frac{1}{\sqrt{2\pi\Delta_\varphi}} \exp\left(-\frac{x^2}{2\Delta_\varphi}\right), \tag{A4}$$

with variance given by

$$\Delta_\varphi = \frac{1}{2}\left[1 + P_\varphi\left(2\sinh^2(r) + \cos[2(\gamma_\varphi - \theta)]\sinh 2r\right)\right]. \tag{A5}$$

The probability distribution of the measurement determines the Fisher information $F(\varphi)$ according to

$$F(\varphi) = \int \mathrm{d}x\, p(x|\varphi)\left(\frac{\partial \ln p(x|\varphi)}{\partial\varphi}\right)^2 = \frac{1}{2}\left(\frac{\partial_\varphi\Delta_\varphi}{\Delta_\varphi}\right)^2. \tag{A6}$$

Recalling that $\sinh^2 r = N$, (A5) reads

$$\Delta_\varphi = \frac{1}{2} + P_\varphi f_\varphi(N), \tag{A7}$$

where

$$f_\varphi(N) = N + \cos[2(\gamma_\varphi - \theta)]\sqrt{N(N+1)}, \tag{A8}$$

and the derivative of (A7) reads

$$\partial_\varphi\Delta_\varphi = (\partial_\varphi P_\varphi)f_\varphi(N) + 2P_\varphi(\partial_\varphi\gamma_\varphi)h_\varphi(N), \tag{A9}$$

where

$$h_\varphi(N) = \sin[2(\gamma_\varphi - \theta)]\sqrt{N(N+1)}. \tag{A10}$$

Substituting (A7) and (A9) into (A6) we finally obtain the expression of the Fisher information:

$$F(\varphi) = 2\left(\frac{(\partial_\varphi P_\varphi)f_\varphi(N) + 2P_\varphi(\partial_\varphi\gamma_\varphi)h_\varphi(N)}{1 + 2P_\varphi f_\varphi(N)}\right)^2. \tag{A11}$$

If the direction of the quadrature being measured satisfies condition (5), the asymptotic behaviour of $f_\varphi(N)$ and $h_\varphi(N)$ for large N is given by:

$$f_\varphi(N) = -\frac{1}{2} + \frac{2k_\varphi^2}{N} + \frac{1}{8N} + O\left(\frac{1}{N^2}\right), \tag{A12}$$

$$h_\varphi(N) = 2k_\varphi\left(1 + \frac{1}{2N}\right) + O\left(\frac{1}{N^2}\right). \tag{A13}$$

Substituting (A12) and (A13) into (A11) and using condition (6), one immediately obtains (7).

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